PHYSICAL REVIEW E VOLUME 56, NUMBER 3 SEPTEMBER 1997

Noise, order, and spatiotemporal intermittency

H. L. Yang,[1,2] Z. Q. Huang,[2] and E. J. Ding[1,2,3]
[1]*China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100 080, China*
[2]*Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China*
[3]*Institute of Theoretical Physics, Academia Sinica, Beijing 100 080, China*

(Received 18 December 1996; revised manuscript received 13 May 1997)

In a large array of globally coupled random bistable units, an ordered phase can appear at an intermediate noise strength. In company with the appearance of the ordered phase, super transients and spatiotemporal intermittencies can be found. The analysis based on a mean-field theory shows that the appearance of a fascinating ordered phase is caused by a phenomenon named the array enhanced tunnel crisis resulting from the nontrivial cooperative effect of the noise, the nonlinearity, and the coupling among units.
[S1063-651X(97)51109-3]

PACS number(s): 05.45.+b

Complex systems, ranging from economic markets and ecosystems to earthquakes and the turbulent flow, have generated a lot of research interest in recent years. The most striking feature of many composite systems containing a large number of elements is that fascinating global phenomena arise out of seemingly simple local dynamics. Furthermore, fluctuations, such as thermal and quantum noises, are intrinsic in dynamical systems. It is then of considerable importance to investigate the influence of noises on such spatially extended systems. Recent results have afford a glimpse into the richness of the behavior that is possible in large arrays of noise coupled oscillators [1–6]. A novel phenomenon named noise-induced nonequilibrium phase transition was reported by Van den Broeck, Parrondo, and Toral [1]. Such a phase transition is characterized by a breaking of ergodicity and the appearance of multiple stable states. This is different from the so-called noise-induced transition [7–9], in which only the shape of a probability density changes qualitatively under the influence of noises. A related work done by Linder *et al.* is about the array enhanced stochastic resonance and the spatiotemporal synchronization [2]. Both of the two groups have shown evidence for the appearance of a noise-induced ordered phase, which cannot be observed in the absence of noises. We attempt to give a direct physical interpretation of the mechanism for this phase transition. In this paper, we try to use a chain of globally coupled random bistable units as a special case to shed some light on this subject. Numerical calculations show that, with increased noise strength, an ordered phase appears at a critical value of the noise strength. In this phase, all the units in the array evolve to a certain one of two coexisting states, while no units are attracted to the other one. With further increased noise strength, this ordered phase is destroyed. Units switch cooperatively between the two states. The time interval between two consecutive switches becomes shorter and shorter with an increase of noise strength, and the system will finally reenter a disorder phase. The analysis based on a kind of mean field theory shows that the ordered phase and the accompanying spatiotemporal intermittency are induced by a phenomenon named the array enhanced tunnel crisis, which results from the cooperative effect between the noise, the nonlinearity, and the coupling among units.

The model we have studied is just a globally coupled random map lattice,

$$y_{n+1}^{(i)}=(1-\epsilon)F(y_n^{(i)},z_n^{(i)})+\epsilon\overline{y_n}, \tag{1}$$

where n is the discrete time step, i is the lattice point index, ϵ represents the coupling strength, $z_n^{(i)}$ can be a random variable influencing the dynamics at site i, and $\overline{y_n}$ is the spatio-average defined by

$$\overline{y_n}=\frac{1}{N}\sum_{i=1}^{N}F(y_n^{(i)},z_n^{(i)})=\frac{1}{N}\sum_{i=1}^{N}y_{n+1}^{(i)}. \tag{2}$$

This type of model might be motivated in part by considering a hypothetical physical situation in which a system consisting of N identical units is embedded in a noisy environment.

For simplicity, we use the randomly shifted piecewise linear map as the local dynamics,

$$F(y_n^{(i)},z_n^{(i)})=f(y_n^{(i)})+z_n^{(i)} \quad (\mathrm{mod}\ 1), \tag{3}$$

where

$$f(y)=\begin{cases}\dfrac{5y}{3} & \text{if}\quad 0\leqslant y<\dfrac{1}{5}\\[2mm] \dfrac{1}{3} & \text{if}\quad \dfrac{1}{5}\leqslant y<\dfrac{2}{5}\\[2mm] \dfrac{5y}{3}-\dfrac{1}{3} & \text{if}\quad \dfrac{2}{5}\leqslant y<\dfrac{3}{5}\\[2mm] \dfrac{2}{3} & \text{if}\quad \dfrac{3}{5}\leqslant y<\dfrac{4}{5}\\[2mm] \dfrac{5y}{3}-\dfrac{2}{3} & \text{if}\quad \dfrac{4}{5}\leqslant y<1,\end{cases} \tag{4}$$

$$z_n^{(i)}=bx_n^{(i)}. \tag{5}$$

b is a positive real constant, $x_n^{(i)}$ is a series of random numbers homogeneously distributed in the interval $[-1,1]$, and it satisfies that $\langle x_n^{(i)}x_m^{(j)}\rangle\sim\delta_{mn}\delta_{ij}$. It can easily be seen that a

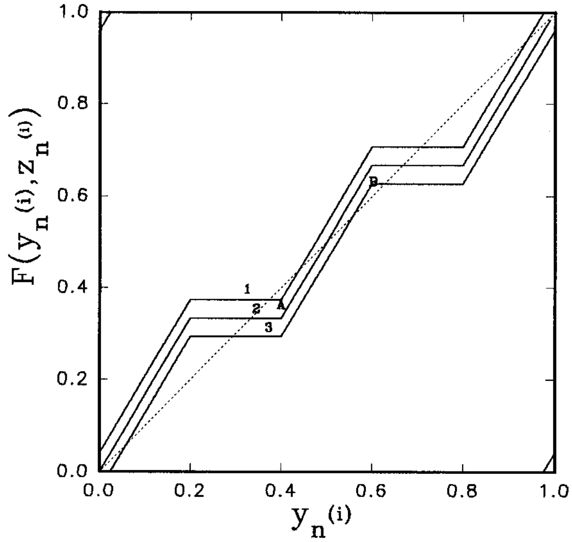

FIG. 1. The piecewise linear map (1) with $\epsilon=0.0$. The three curves are calculated from 1, $y_{n+1}=f(y_n)$; 2, $y_{n+1}=f(y_n)+b$; 3, $y_{n+1}=f(y_n)-b$, respectively.

single piecewise linear map (3) has two coexisting stable states $y_1^s=\frac{1}{3}$ and $y_2^s=\frac{2}{3}$ without the influence of the random noise (see Fig. 1). With an increase of noise strength, two states are blurred into two bands correspondingly. The map function (3) will be tangent with the 45° line occasionally at point A (or B) when the noise strength reaches a critical value $b_c=\frac{1}{15}$. After that, a narrow tunnel appears somewhere between the map function and the 45° line. Then, phase points are able to switch between two bands randomly. This is the ''tunnel crisis'' reported in Ref. [10]. Without loss of generality, we will use the case of coupling strength $\epsilon=0.2$ as an example to show you below what happens when a large array of such units are coupled globally together.

Numerical calculations show that for randomly selected initial conditions three different types of behaviors can be found in globally coupled units during a slow increase of noise strength. For weak noise strengths, units evolve to a certain ''preselected'' band and stay there forever, just as they do not feel the coupling among them [see Fig. 2(a)]. We call this behavior the quenched disordered phase, since once a unit evolves to a certain band it stays there forever and a random selection of initial conditions makes each unit evolving to one of two different bands at random. For the numbers of units evolving to the two bands are equal, this phase is a spatiohomogeneous one with a spatial average $\overline{y_n}=0.5$. For an intermediate noise strength, units starting from random initial conditions first evolve into two groups and form two bands respectively. After a long period of two-band evolution, units in a certain band jump one by one into the other band till all units in the whole array are in the same band. After that, all units are confined to this band without spreading or switching to the other band [see Fig. 2(b)]. In this phase we have the spatial average $\overline{y_n}=\frac{1}{3}$ or $\overline{y_n}=\frac{2}{3}$ for different initial conditions. It is the symmetry-breaking ordered phase. With further stronger noise strengths, the ordered phase also loses its stability. After a period of transient process, the whole array evolves into one band just as in the ordered phase. But all units cannot stay in this band forever. Instead, they will switch between two bands randomly. This is the phase named spatiotemporal intermittency [see Fig. 2(c)]. It does not have a stationary spatial average $\overline{y_n}$: in case the array switches from one band to the other, $\overline{y_n}$ will change its value correspondingly. The time interval between two consecutive switches becomes shorter and shorter with the increase of noise strength. For a large enough noise strength b, all units will switch between the two bands very frequently and even spread to cover them. The system enters a disordered phase again.

It can easily be seen that the map function (4) is symmetrical with respect to the point $(\frac{1}{2},\frac{1}{2})$, i.e., $\frac{1}{2}+f(y-\frac{1}{2})=\frac{1}{2}-f(\frac{1}{2}-y)$. In addition, initial conditions are homogeneously distributed in the interval [0,1]. Why can a symmetry-breaking ordered phase appear in this system for an intermediate noise strength? In order to clarify the mechanism, we shall just consider a single unit in the array and view the influence of other ones on it as a random environment [1]. Then, the map (1) can be transformed into

$$y_{n+1}=f(y_n)+w_n, \tag{6}$$

where

$$w_n=z_n(1-\epsilon)+\epsilon[\overline{y_n}-f(y_n)]. \tag{7}$$

Since only one site is considered, here and below, all the superscripts are omitted for simplicity. The map (6) is just of the same form as the randomly shifted piecewise linear map (3). The only difference is that the ''noise'' w_n here consists of two terms. One is $z_n(1-\epsilon)$, which is of determined strength $b(1-\epsilon)$, just as z_n in map (3). The other term is $\epsilon[\overline{y_n}-f(y_n)]$ whose strength varies with the deviation $\overline{y_n}-f(y_n)$. We would like to call it the ''feedback noise.'' It will be shown below that it is the ''feedback noise'' that brings about the symmetry-breaking ordered phase and the spatiotemporal intermittency.

For large enough noise strength, units starting from randomly selected initial conditions homogeneously distributed in the interval [0,1] first evolve into two groups and form two bands respectively. Since the numbers of points evolving to the two bands are equal statistically, the spatial average of the variable y_n is $\overline{y_n}=\frac{1}{2}$. For a point in the lower band [11], the driving ''noise'' w_n is of strength $b(1-\epsilon)+\epsilon(\frac{1}{2}-\frac{1}{3})$. Just as the case in map (3), the map function (6) may tangent occasionally with the 45° line when the noise strength satisfies the condition

$$b(1-\epsilon)+\epsilon(\tfrac{1}{2}-\tfrac{1}{3})=\tfrac{1}{15}. \tag{8}$$

It gives a critical curve in the (b,ϵ) plane. After that, a narrow tunnel appears occasionally between the map (6) and the 45° line. Then, phase points can jump from one band to the other one occasionally. We call this jump the array enhanced tunnel crisis. It provides for the probability of phase points to jump from one band to the other. For parameters above the curve (8), the quenched disordered phase is destroyed by the array enhanced tunnel crisis. In case of statistic fluctuations, the numbers of points in the two bands may become unequal after a period of jumping. Then, the spatial

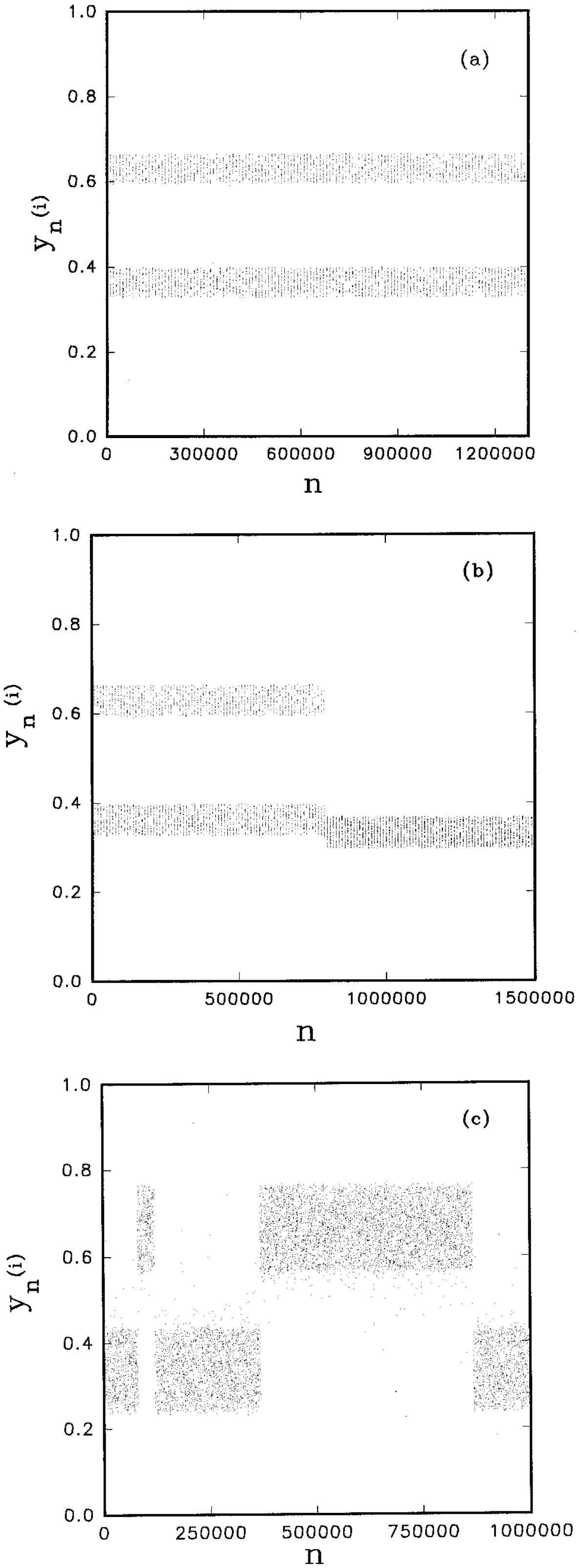

FIG. 2. The temporal evolution of the random map lattices (1). Abscissa is iteration step n and ordinate is variable $y_n^{(i)}$. Values of $y_n^{(i)}$ for all the units in the array have been plotted in the same figure. The parameter setting is $\epsilon=0.2$, $N=16$, and $B=15b$ here. (a) The quenched disordered phase for $B=0.35$ (plotted every 10 000th step); (b) The symmetry-breaking ordered phase for $B=0.9$ (plotted every 1000th step); (c) The spatiotemporal intermittency for $B=1.8$ (plotted every 1000th step).

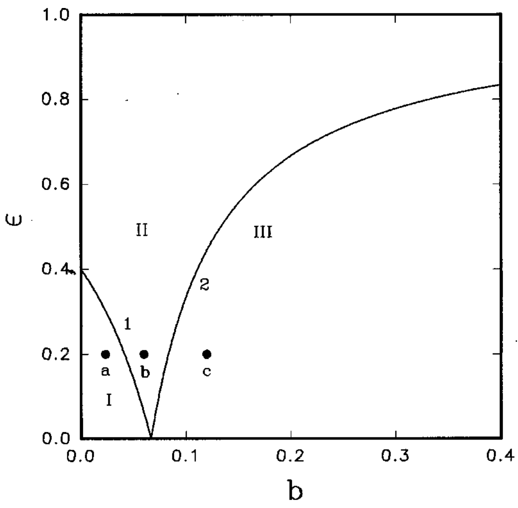

FIG. 3. The phase diagram of the coupled random map lattice (1). The two curves 1, 2 are calculated from Eqs. (8) and (11) respectively. The three phases are I the quenched disordered phase, II the symmetry-breaking ordered phase, III the spatiotemporal intermittency phase. Just above curve 1, a super long two-band transient iteration can be observed. The three points denoted by a, b, and c correspond to the cases plotted in Fig. 2.

average of y_n will not be $\frac{1}{2}$ now. If we denote the fraction of points in the lower band as n_L, the spatial average is

$$\overline{y}=\tfrac{1}{3}n_L+\tfrac{2}{3}(1-n_L)=\tfrac{1}{2}-\tfrac{1}{3}(n_L-\tfrac{1}{2}). \tag{9}$$

By inserting Eq. (9) into Eq. (7), the strength b_f of the ''feedback'' term in w_n can be obtained,

$$b_f=\begin{cases}\dfrac{1}{6}-\dfrac{1}{3}\left(n_L-\dfrac{1}{2}\right) & \text{for} \quad f(y_n) \text{ in the lower band}\\[2ex] \dfrac{1}{6}+\dfrac{1}{3}\left(n_L-\dfrac{1}{2}\right) & \text{for} \quad f(y_n) \text{ in the upper band.}\end{cases} \tag{10}$$

It is different for phase points in the two bands (see Fig. 3). Then, jumping of phase points between two bands in two directions cannot be symmetric now. Without loss of generality, we assume that the number of points in the upper band is larger than that in the lower band, i.e., $n_L<\frac{1}{2}$. The strength of the ''noise'' w_n for points in the lower band is $b(1-\epsilon)+\epsilon[\frac{1}{6}+\frac{1}{3}(\frac{1}{2}-n_L)]$, which is greater than the strength $b(1-\epsilon)+\epsilon[\frac{1}{6}-\frac{1}{3}(\frac{1}{2}-n_L)]$ for points in the upper band. It is obvious that, for a stronger noise, the tunnel, between the map function and the 45° line has a larger probability to appear and have a larger width. As a result, points in the lower band can jump more easily to the upper band via the tunnel, while jumping of points in the upper band to the lower band is more difficult. This difference of jumping probably in two directions gives a net flow of phase points from the lower band to the upper band. Appearance of the net flow makes the difference in probability of jumping in two directions more acute, and even no points can jump from the upper band to the lower band for small enough n_L. Finally, all the units are in the upper band. This is the symmetry-breaking ordered phase. In this phase, the strength of ''noise'' w_n is $b(1-\epsilon)$. Since $b(1-\epsilon)$ is smaller than $\frac{1}{15}$ for not very strong noise, no tunnel can appear between the map function (6) and the 45° line. So, once the whole array has entered a certain band, phase points will confined in that band. In other words, the ordered phase we get here is a structurally stable one.

For even stronger noise, the ordered phase becomes unstable when $b(1-\epsilon)>\frac{1}{15}$. The critical values of parameters form the other curve

$$b(1-\epsilon)=\frac{1}{15} \tag{11}$$

in the phase plane (b,ϵ). For parameters above this curve, phase points will switch randomly between the two bands instead of being confined in one band. The reason is that, even for the case that all the units lie in the same band, a tunnel can appear between the map function (6) and the 45° line now. Then, there may be points escaping from the band where they originally stayed. After the first point's escaping, escape of other ones becomes a little easier and easier. Due to this accelerating effect, more and more points escape. The appearance of points in the other band is due to statistic fluctuations. Once the fraction of points in the other band is greater than $\frac{1}{2}$, the net flow will reverse its direction. Now, phase points tend to jump to this band and stay there. As a result, all the units in the array jump to the band eventually. And so on, all the units in the array switch between the two bands coherently.

Numerical calculations show that statistic average of the time interval τ between two consecutive switches behaves as

$$\langle\tau\rangle\sim\exp(N\delta^{-\gamma}), \tag{12}$$

where N is the system size, $\delta=b-b_c$ is the deviation of the noise strength from its critical value and γ is a critical exponent. For our model the critical exponent $\gamma=0.5$. Similar behavior for the length of the two-band transient process can be found just beyond the critical curve (8).

Finally, we make some remarks below as a conclusion. First, in a chain of globally coupled random units, a symmetry-breaking ordered phase and accompanying spatiotemporal intermittency can be observed for a intermediate noise strength. Second, this transition possesses some features of the first-order phase transition such as the sudden appearance of a nonzero ordered-parameter, which is $|\overline{y}-\frac{1}{2}|$ in our model. On the other hand, near critical curves (8) and (11), perfect scaling relations such as (12) can be found which is the characteristic feature of a second-order transition. And the appearance of the intermittency behavior is a characteristic feature of a dynamic system. All these indicate that the transition to an ordered phase in our model is a purely nonequilibrium phenomenon. Third, theoretical analysis and numerical calculations have shown that the ordered phase and the spatiotemporal intermittency observed are not transient phenomena. Finally, we would like to presume that the mechanism of the transition here is the same as that in Ref. [1], since the mean field coupling among units, which brings about the ''feedback noise'' in the effective model, is the common feature of the two systems. The difference between the two cases is that the system presented in this paper has two coexisting state without the noise, while the two coexisting states of the system in Ref. [1] are induced by external noise.

This project is supported by the National Nature Science Foundation, the National Basic Research Project ''Nonlinear Science,'' the Youth Foundation of BNU, and the Educational Committee of the State Council through the Foundation of Doctoral Training.

[1] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994); C. Van den Broeck, J. M. R. Parrondo, J. Armero, and A. Hernández Machado, Phys. Rev. E **49**, 2639 (1994), and references therein.
[2] J. F. Linder, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, Phys. Rev. Lett. **75**, 3 (1995).
[3] P. Jung, U. Behn, E. Pantazelou, and F. Moss, Phys. Rev. A **46**, R1709 (1992); P. Jung and G. Mayer-kress, Phys. Rev. Lett. **74**, 2130 (1995).
[4] K. Wiesenfeld, P. Colet and S. H. Strogatz, Phys. Rev. Lett. **76**, 404 (1996); Y. Braiman, J. F. Linder, and W. L. Ditto, Nature (London) **378**, 465 (1995); Y. Braiman, W. L. Ditto, K. Wiesenfeld, and M. L. Spano, Phys. Lett. A **206**, 54 (1995).
[5] J. P. Crutchfield and K. Kaneko, Phys. Rev. Lett. **60**, 2715 (1988); K. Kaneko, Phys. Lett. A **149**, 105 (1990).
[6] G. Hu, H. Haken and F. Xie, Phys. Rev. Lett. **77**, 1925 (1996).
[7] *Noise in Nonlinear Dynamical Systems*, edited by F. Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, 1989).
[8] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer-Verlag, Berlin, 1984).
[9] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).
[10] H. L. Yang, Z. Q. Huang, and E. J. Ding, Phys. Rev. Lett. **77**, 4899 (1996).
[11] Calculations for a point in the upper band can give the same critical curve.